

# A QUANTITATIVE OBSTRUCTION TO COLLAPSING SURFACES

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## 1. INTRODUCTION

S. Alesker posed the following question at MathOverflow [1]. Let  $(M_i)$  be a sequence of 2-dimensional orientable closed surfaces of genus  $g$  with smooth Riemannian metrics with the Gaussian curvature at least  $-1$  and diameter at most  $D$ . By the Gromov compactness theorem, one can choose a subsequence converging in the Gromov–Hausdorff sense to a compact Alexandrov space with curvature at least  $-1$  and Hausdorff dimension 0, 1, or 2. Let us assume that the limit space has dimension 1. Then it is either a circle or a segment. Can these possibilities (circle and segment) be obtained in the limit?

## 2. SOLUTION

The solution exploits Gromov’s notion of the filling radius of a manifold [2] but the argument is elementary and is reproduced below together with all the necessary background, relying only on basic Jacobi field estimates and basic homotopy theory.

The *systole* is the least length of a noncontractible loop of the surface  $M$ . The *filling radius* of  $M$  is defined as the infimum of all  $\epsilon > 0$  such that the inclusion of  $M$  in its  $\epsilon$ -neighborhood in any strongly isometric embedding of  $M$  in a Banach space sends the fundamental homology class of  $M$  to zero in the sense of the induced homomorphism on  $H_2(M)$ . Here the embedding can be taken to be to the space of bounded functions on  $M$  which sends a point of  $M$  to the distance function from that point. This embedding is strongly isometric if the function space is equipped with the sup-norm.

**Lemma 2.1** (Gromov’s lemma). *The systole of a hyperbolic surface  $M$  is at most six times the filling radius of  $M$ .*

*Proof.* Consider a strongly isometric embedding of the surface  $M$  into a Banach space  $B$  which can be assumed finite-dimensional by [3]. Suppose  $M$  is “filled” (in the homological sense) by a chain  $C$  (in the sense

that  $M$  is the boundary of  $C$ ), so that the fundamental homology class of  $M$  vanishes under the induced homomorphism  $H_2(M) \rightarrow H_2(C)$ .

Consider a triangulation of  $C$  into infinitesimal triangles (here the term “infinitesimal” is used informally in its meaning “sufficiently small” though this can actually be rendered rigorous). Suppose  $C$  is contained in an open  $R$ -neighborhood of  $M$  in  $B$  where  $R > 0$  is less than a sixth of the systole. We will now retract  $C$  back to  $M$ , while fixing the subset  $M \subseteq C$ . This would contradict the nonvanishing of the fundamental class.

For each vertex of the triangulation, we choose a nearest point of  $M$ . To extend the retraction to the 1-skeleton of  $C$ , map each edge to a minimizing path of length less than  $2R$  joining the images of the two vertices in  $M$ . Hence the boundary of each 2-cell of the triangulation is sent to a loop of length less than  $6R$ . Since this is less than the systole, the map can now be extended to the 2-skeleton of  $C$ .

One might have thought that there is a lot of work left to be done to extend the map to the 3-skeleton. However, the universal cover of  $M$  is contractible and therefore  $\pi_2(M) = 0$ . Therefore the retraction extends to all of  $C$ . The contradiction completes the proof of the lemma.  $\square$

To prove a suitable lower bound so as to rule out positive-codimension collapse, consider a noncontractible closed geodesic  $\gamma \subseteq M$  of length equal to the systole  $\text{sys} = \text{sys}(M)$ , and the normal exponential map along  $\gamma$ . Using the lower curvature bound, we obtain an upper bound on the total area of  $M$  as  $\text{sys} \cdot \sinh(D)$  where  $D$  is the diameter. The bound follows by applying Rauch bounds on Jacobi fields (this is an ingredient in the proof of Toponogov’s theorem). Therefore the systole is bounded below by

$$\frac{\text{area}}{\sinh D}.$$

Meanwhile the area is bounded below by the Gauss–Bonnet theorem:

$$\text{area}(M) \geq - \int_M K = 2\pi(2g - 2)$$

where  $g$  is the genus. Furthermore the filling radius of  $M$  is bounded below by a sixth of the systole by Gromov’s Lemma 2.1. The least Gromov–Hausdorff distance from  $M$  to a graph is bounded below by the filling radius. This proves that hyperbolic surfaces of curvature bounded below by  $-1$  with diameter bounded above by  $D$  cannot collapse, so that a Gromov–Hausdorff limit is necessarily 2-dimensional. Note that we obtain a quantitative lower bound, rather than merely the nonexistence of Shioya–Yamaguchi-type collapse [5], [4] to spaces of positive codimension.

## REFERENCES

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See <http://mathoverflow.net/questions/236001>
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